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LETTER TO THE EDITOR

The Liouville theory in the degenerate representation of the Virasoro algebra

Toshiya Kawai and Kengo Yamagishi
Institute of Physics, University of Tokyo-Komaba, Megro-ku, Tokyo 153, Japan

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Abstract. An attempt is made to extend the results of Gervais and Neveu to the degenerate representation at an arbitrary level N of the Virasoro algebra in the Liouville theory. As an application, the Liouville string theory (in D spacetime dimensions) is considered. The tachyonic states appear under the consistency condition of the theory: $D \leq 1$.

Since the new formulation of the string theory initiated by Polyakov (1981), various analyses have been carried out based on the (quantum) Liouville theory in $1+1$ spacetime dimensions (Curtright and Thorn 1982, Braaten *et al* 1982, 1983, 1984, Gervais and Neveu 1982a, b, 1983). Recently, some exact results have been obtained by Gervais and Neveu (1984a, b) for Green functions of the Liouville field e^{φ} (a kind of vertex operator) by solving the linear second-order differential equations. The analysis has been carried out based on the conformal symmetry which is produced by an infinite number of generators L_n ($n \in \mathbb{Z}$) satisfying the Virasoro algebra with a central charge at the classical level. From the algebraic point of view, the previous results (Gervais and Neveu 1984a, b) will be explained as being a specific realisation of this infinite algebra. The central charge is usually considered as the obstruction for this in the Hilbert space. However, this can be circumvented by demanding the conditions $L_n|\text{phys}\rangle = 0$ ($n > 0$). The central charge is no problem in this case.

However, many questions and problems remain unanswered in their approach: is the solution unique? What is the meaning of their constraint equation $2\hbar\eta^2 - \eta + 1 = 0$? Is it possible by choosing other solutions to make the dimensions ($=D$) greater than one for the embedding spacetime of the string?

In this letter we attempt to extend their formalism, referring to the conformal bootstrap machinery developed recently by Belavin *et al* (1984a, b), Friedan *et al* (1984), Dotsenko (1984) and Dotsenko and Fateev (1984). In the language of the latter program the analysis of Gervais and Neveu (1984a, b) is restricted to the conformally covariant operator degenerate at level two. Here, we proceed further and derive explicitly the linear differential equation for Green functions including level-three operators. The connection with the Kac formula (Kac 1979, 1982, Feigin and Fuks 1982, 1983) for the anomalous dimension is noted. A brief analysis is also carried out for the application of the results obtained here to the Liouville string theory for the degenerate representation at an arbitrary level N. Tachyonic states appear under the consistency condition of the theory. The notation is the same as in Gervais and Neveu (1984a, b).

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Let us start by considering the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + [n^3/4\hbar + \tfrac{1}{12}(n^3-n)]\delta_{n+m}, \tag{1}$$

where the operator L_n is defined by

$$L_n = (2\hbar)^{-1}\colon\left(\sum_{r=-\infty}^{\infty} P_r P_{n-r} - \mathrm{i}nP_n\right)\colon \tag{2}$$

with the commutation relation

$$[p_n, p_m] = n\hbar\delta_{n+m}. \tag{3}$$

According to Gervais and Neveu (1984a, b), the algebraic system (1)-(3) is connected to the quantum Liouville theory by the following correspondence. The Liouville theory defined by the Lagrangian

$$\mathcal{L} = (1/16\pi\hbar)[\tfrac{1}{2}(\partial\varphi)^2 - \mathrm{e}^{\varphi}] \tag{4}$$

has conformal symmetry. Its generator is given as usual by the conformally improved energy-momentum tensor $\Theta^{0\mu}$ ($\mu = 0, 1$). Because of the geometric origin of (4) this quantity can be rewritten by the Schwarzian derivative† (Flanders 1971, Hille 1976) as

$$\Theta^{00} = -2\hbar^{-1}(\{A, \sigma\} + \{B, \sigma\}), \tag{5}$$

$$\Theta^{01} = -2\hbar^{-1}(\{A, \sigma\} - \{B, \sigma\}). \tag{6}$$

Here A and B are dynamical variables introduced by

$$\varphi(\tau, \sigma) = \ln[-8A_\sigma B_\sigma/(A-B)^2] \tag{7}$$

instead of $\varphi(\tau, \sigma)$. The advantage of making use of these new variables is that the mode expansion becomes simpler compared with the original nonlinear field $\varphi(\tau, \sigma)$. Defining

$$P \equiv -\tfrac{1}{2}\partial_\sigma \ln A_\sigma, \qquad Q \equiv -\tfrac{1}{2}\partial_\sigma \ln B_\sigma, \tag{8}$$

and expanding (8) as

$$P \equiv \sum_{n=-\infty}^{\infty} p_n \exp(-\mathrm{i}n\sigma), \qquad Q \equiv \sum_{n=-\infty}^{\infty} q_n \exp(-\mathrm{i}n\sigma) \tag{9}$$

Gervais and Neveu have derived the previous algebras by the correspondence‡

$$\Theta^{00} \pm \Theta^{01} = 16 \sum_n L_n^{(\pm)} \exp(-\mathrm{i}n\sigma). \tag{10}$$

In terms of the new variables (8) and (9), the conformal covariant Liouville operator $\mathrm{e}^{-\varphi/2}$ has been decomposed into more elementary ones

$$\mathrm{e}^{-\varphi/2} = \mathrm{i}(\psi_2\chi_1 - \psi_1\chi_2), \tag{11}$$

where ψ_i, χ_i take the form similar to the vertex operator (Schwarz 1973, Scherk 1975)

† The Scwarzian is defined as

$$\{A(z), z\} \equiv \partial_z(A_{zz}/A_z) - (A_{zz}/A_z)^2/2,$$

where A_z means $\partial_z A$.

‡ In equations (1)-(3), superscripts ($\pm$) are suppressed for brevity. $L_n^{(+)}$ and $L_n^{(-)}$ commute with each other.

in the dual model as

$$\psi_1(\sigma)=\exp(-i\hbar\eta^2\sigma/2)\exp(x_0\eta)\exp(p_0\eta\sigma):\exp\left(i\eta\sum_{m\neq 0}\frac{p_m}{m}\exp(-im\sigma)\right):,$$
$$\psi_2(\sigma)=\psi_1^+(\sigma),\qquad [x_0,p_0]=i\hbar \tag{12}$$

and satisfy† the conformal covariance relation

$$[L_n,\psi_i(\sigma)]=-i\exp(in\sigma)(\partial/\partial\sigma+in\delta)\psi_i(\sigma). \tag{13}$$

The conformal weight δ equals $-(\eta+\hbar\eta^2)/2$. The most interesting observation made by Gervais and Neveu is that the quantities ψ_i satisfy the closed triangular algebraic relation‡, provided the constraint equation

$$2\hbar\eta^2-\eta+1=0 \tag{14}$$

is satisfied.

Let us first consider the meaning of equation (14), which remains obscure in Gervais and Neveu (1984a, b). A straightforward calculation leads to the following linear differential equation

$$(\partial_\sigma+\tfrac{1}{2}i\hbar\eta^2)^2\langle\omega_f|\psi_1(\sigma)-2\hbar\eta^2\langle\omega_f|\psi_1(\sigma)\left(\sum_{m\geqslant 0}\exp(-im\sigma)L_m\right)$$
$$=-i\eta(2\hbar\eta^2-\eta+1)\langle\omega_f|\psi_1(\sigma)\left(\sum_{m\geqslant 0}\exp(-im\sigma)mp_m\right). \tag{15}$$

Here $\langle\omega_f|$ is the final state satisfying the following conditions

$$L_0|\omega_f\rangle=-(\omega_f^2/4\pi h)|\omega_f\rangle,\qquad L_n|\omega_f\rangle=0\qquad(n>0). \tag{16}$$

The vanishing of the RHS of (15) ensures the closure of the differential equation for arbitrary correlation functions with the help of the physical state conditions

$$L_n|\text{phys}\rangle=0\qquad(n>0). \tag{17}$$

The second-order differential equation (15) reminds us of the fact that correlation functions, including conformal operators degenerate at the second level, obey a similar rule in the conformal bootstrap program for the statistical system in two dimensions at the critical point (Belavin *et al* 1984a, b, Dotsenko 1984, Dotsenko and Fateev 1984). Representing the conformal weight δ in terms of the central charge $\frac{1}{12}C=\frac{1}{12}+(4\hbar)^{-1}$, we can show that equation (14) is exactly the condition that fixes the level of the operator $\psi_i(\chi_i)$ to two, namely,

$$\delta=\{5-C\pm[(C-1)(C-25)]^{1/2}\}/16=\Delta_{2,1}\text{ or }\Delta_{1,2}, \tag{18}$$

where $\Delta_{k,s}$ is the Kac formula (Kac 1979, 1982, Feigin and Fuks 1982, 1983) defined by

$$\Delta_{k,s}=\{(13-C)(k^2+s^2)\pm(k^2-s^2)[(C-1)(C-25)]^{1/2}-24ks-2+2C\}/48. \tag{19}$$

For an arbitrary $\delta=\Delta_{k,s}$ (level $N=ks$), on the other hand, the corresponding constraint equation (like (14)) is expected to be

$$2\hbar\eta^2-(k+s-2)\eta+(k-s)^2+(k-1)(s-1)/2\hbar=0. \tag{20}$$

† χ_i's are given by the same form with p_n replaced by q_n. The c-number η is determined below.
‡ See Gervais and Neveu (1984a) for its explicit form.

Indeed, we can confirm these correspondences at level three by further differentiating equation (15). The result becomes

$$\partial_\sigma\left[(\partial_\sigma+\tfrac{1}{2}\mathrm{i}\hbar\eta^2)^2\langle\omega_{\mathrm{f}}|\psi_1(\sigma)-2\hbar\eta^2\langle\omega_{\mathrm{f}}|\psi_1(\sigma)\left(\sum_{m\geq 0}\exp(-\mathrm{i}m\sigma)L_m\right)\right]$$

$$+\mathrm{i}\hbar\eta^2(2\hbar\eta^2-\eta+1)\langle\omega_{\mathrm{f}}|\psi_1(\sigma)\left(\sum_{m\geq 0}m\exp(-\mathrm{i}m\sigma)L_m\right)$$

$$=-\eta(2\hbar\eta^2-\eta+1)(\tfrac{1}{2}\hbar\eta^2-\tfrac{1}{2}\eta+1)$$

$$\times\langle\omega_{\mathrm{f}}|\psi_1(\sigma)\left(\sum_{m\geq 0}m^2\exp(-\mathrm{i}m\sigma)p_m\right). \tag{21}$$

From (20) we can recognise immediately that vanishing of the RHS of equation (21) occurs for values of η such that the level of ψ_1 equals three (or needless to say, two or one). This is precisely the rule for the correlation function including level-three operators. (Generally, the Nth-order linear differential equation for the operators degenerate at level N (Belavin *et al* 1984a, b).) We reasonably anticipate that these correspondences remain true to an operator of arbitrary level by the repeated differentiation of the equation (21).

Here some comments are in order.

(1) In the RHS of equation (21) (and (15)), there appears the Kac determinant at the third (second) level. We may conjecture that explicit use of the vertex operator induces the Kac determinant in the course of deriving the closed-form differential equation for correlation functions†. (Needless to say, equation (15) ((21)) (setting the RHS to zero) is precisely the condition that the representation be degenerate at level two (three), i.e., $w=0$ where w is the singular vector at level two (three) in the Verma module over the Virasoro algebra (Feigin and Fuks 1982, 1983).)

(2) We may reverse the arguments and expect the results obtained here from the beginning, since the conformal bootstrap machinery for the statistical system can be applied to the present problem with a slight modification. According to Dotsenko and Fateev (1984), the correlation functions of the degenerate conformal theory are represented by the integrals of average over the vertex operators‡ $V_\alpha = :\exp(\mathrm{i}\alpha\varphi(z)):$ in a Coulomb-like system with a fixed charge $-2\alpha_0$ placed at infinity. The corresponding energy-momentum tensor is

$$T(z)=-\tfrac{1}{4}:\partial_z\varphi\partial_z\varphi:+\mathrm{i}\alpha_0\partial_z^2\varphi. \tag{22}$$

Comparing (21) with (2), we find that they are essentially the same except that (2) corresponds to an imaginary $\alpha_0(\hbar^{-1}\sim-\alpha_0^2)$. (In fact, from the purely algebraic point of view, we may take any complex number of α_0, accordingly $C, \Delta_{k,s}\in\mathbb{C}$ as far as deciding a special degenerate representation for the conformal algebra.) Therefore, we could employ the techniques of Belavin *et al* (1984a, b) and Dotsenko (1984) to derive equation (15) ((21)) (with the RHS setting to zero).

(3) Because of the imaginary α_0 the central charge C and the conformal dimension $\Delta_{k,s}$ take the value $C>25$ and $\Delta_{k,s}<0$. This is to be contrasted with the cases in the conformal statistical systems where $0<C<1$ and $\Delta_{k,s}>0$, determined from the physical requirement.

† After finishing the main part of this work we have learned that Thorn (1984) has succeeded in rederiving the Kac determinant completely using the vertex operators in somewhat different manner.

‡ The difference in the quantisation schemes is not essential.

(4) Following the general rule in the degenerate representation, the eigenvalue of L_0 for the state $|\omega_f\rangle$ in (16) should be taken to be†

$$\Delta_{k,s} - (8\hbar)^{-1}, \tag{23}$$

since $|\omega_i\rangle$ ($\langle\omega_f|$) corresponds to the highest weight state $\phi(0)|0\rangle$ ($\langle 0|\phi(\infty)$) in the conformal bootstrap approach.

(5) Belavin *et al* (1984a, b) have noted that for a (finite) conformal transformation $z \mapsto w(z)$, the energy-momentum tensor $T(z)$ of the general conformal theory transforms as

$$T(z) = T(w)(\mathrm{d}w/\mathrm{d}z)^2 + C\{w, z\}/12, \tag{24}$$

which is tantamount to the Virasoro algebra. The Schwarzian derivative $\{w, z\}$ has many interesting properties (see e.g. Flanders 1971, Hille 1976) among which are

$$\{w, z\} = 0 \Leftrightarrow w = (az+b)/(cz+d), \qquad ad - bc \neq 0, \tag{25a}$$

$$\{w, z\} = -\{z, w\}(\mathrm{d}w/\mathrm{d}z)^2 \tag{25b}$$

$$\{w, z\} = \{w, \zeta\}(\mathrm{d}\zeta/\mathrm{d}z)^2 + \{\zeta, z\}. \tag{25c}$$

Equation (25*a*) implies that $T(z)$ transforms like a tensor (density) for projective transformations (or the vanishing of the central term for the Virasoro algebra). Equations (25*b*, *c*) ensure that $T(z)$ behaves consistently under the inversion and composition of conformal transformations. In addition, equations (24) and (25*c*) tell us that we have a central charge in the Virasoro algebra at the classical level provided that the (classical) energy-momentum tensor is equal (up to a multiplicative constant) to a Schwarzian derivative as in the present problem‡.

Finally we consider the physical consequences of the operators of higher level in connection with the string theory. The relation between the dimension D of the embedding spacetime of a string and the constant $\hbar$ is independent of the η variable, and is given by

$$25 - D = 3/\hbar. \tag{26}$$

The condition $0 < \hbar \leq \frac{1}{8}$, ensuring the reality of η, is satisfied if $D \leq 1$ as before.

The mass of the lightest scalar particle from the ordinary vertex with Liouville mode (Gervais and Neveu 1984a, b) is

$$M^2 = -\alpha_0 - \omega_i^2/4\pi\hbar. \tag{27}$$

Here α_0 is the intercept corresponding to the ordinary string mode, while $-\omega_i^2/4\pi\hbar$ is the eigenvalue of L_0 for the initial state of the Liouville theory. They are given by

$$\alpha_0 = 1 - (8\hbar)^{-1} \tag{28}$$

and equation (23), respectively. In the naive application of the preceding result (23) the condition for the no tachyonic state $M^2 \geq 0$ is never saturated for an arbitrary set of (k, s). This corresponds to the unitarity problem ($C > 25$, $\Delta_{k,s} < 0$) mentioned before, and will need further consideration.

† This may give rise to problems for unitarity.

‡ If the central term in (24) is of purely quantum origin and the classical energy-momentum tensor transforms as $T(z) = T(w)(\mathrm{d}w/\mathrm{d}z)^2$, in view of equation (25*c*) the quantum corrections to the energy-momentum tensor might be of the form $c\{F, z\}/12$ where F is an analytic function of z.

From these consequences we may conclude that the consistent application of the degenerate representation to the Liouville theory yields a solution for the exact Green function, although some problems remain open concerning unitarity.

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Note. After the completion of this paper, we received a preprint by Gervais and Neveu (1984c) in which related subjects are studied.

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